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Hessian spectrum at the global minimum of high-dimensional random landscapes

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Abstract. Using the replica method we calculate the mean spectral density of the Hessian matrix at the global minimum of a random $N \gg 1$ dimensional isotropic, translationally invariant Gaussian random landscape confined by a parabolic potential with fixed curvature $\mu > 0$. Simple landscapes with generically a single minimum are typical for $\mu > \mu_c$, and we show that the Hessian at the global minimum is always *gapped*, with the low spectral edge being strictly positive. When approaching from above the transitional point $\mu = \mu_c$ separating simple landscapes from "glassy" ones, with exponentially abundant minima, the spectral gap vanishes as $(\mu - \mu_c)^2$. For $\mu < \mu_c$ the Hessian spectrum is qualitatively different for 'moderately complex' and 'genuinely complex' landscapes. The former are typical for short-range correlated random potentials and correspond to 1-step replica-symmetry breaking mechanism. Their Hessian spectra turn out to be again gapped, with the gap vanishing on approaching μ_c from below with a larger critical exponent, as $(\mu_c - \mu)^4$. At the same time in the "most complex" landscapes with long-ranged power-law correlations the replica symmetry is completely broken. We show that in that case the Hessian remains gapless for all values of $\mu < \mu_c$, indicating the presence of 'marginally stable' spatial directions. Finally, the potentials with *logarithmic* correlations share both 1RSB nature and gapless spectrum. The spectral density of the Hessian always takes the semi-circular form, up to a shift and an amplitude that we explicitly calculate.

1. Introduction

1.1. Formulation of the problem

Understanding statistical structure of stationary points (minima, maxima and saddles) of random landscapes and fields of various types and dimensions is a rich problem of intrinsic current interest in various areas of pure and applied mathematics [1–10]. It also keeps attracting steady interest in the theoretical physics community, and this over more than fifty years [11–20], with recent applications to statistical physics [15–17, 19–23], neural networks and complex dynamics [8, 9, 24], string theory [25, 26] and cosmology [27, 28].

One of the most rich, generic and well-studied models in this class is described by the random function

$$\mathcal{H}(\mathbf{u}) = \frac{\mu}{2} \sum_{k=1}^N u_k^2 + V(u_1, \dots, u_N), \quad (1)$$

where the curvature $\mu > 0$ of the non-random confining parabolic potential is used to control the number of stationary points and $V(\mathbf{u})$ is a mean-zero Gaussian-distributed random potential of $\mathbf{u} \in \mathbb{R}^N$ characterized by a particular (rotationally and translational invariant) covariance structure:

$$\overline{V(\mathbf{u}_1)V(\mathbf{u}_2)} = N B \left(\frac{1}{2N} (\mathbf{u}_1 - \mathbf{u}_2)^2 \right), \quad (2)$$

In Eq.(2) and henceforth the notation $\overline{\cdots}$ stands for the quantities averaged over the random potential, and $B(u)$ is a function of order unity belonging to the so-called class \mathcal{D}_∞ described in detail, e.g., in the book by Yaglom [29]. The functions $B(q) \in \mathcal{D}_\infty$ are such that they represent covariances of an isotropic random potential for *any* spatial dimension $N \geq 1$ (see a more detailed discussion below).

The mean number of stationary points and the mean number of minima for the random function $\mathcal{H}(\mathbf{u})$ (which will be referred to as a 'landscape') was investigated in the limit of large dimension $N \gg 1$ in [16] and [19, 20], see also [2, 18, 28]. In particular, it was found that a sharp transition occurs from a 'simple' landscape for $\mu > \mu_c = \sqrt{B''(0)}$ with typically only a single stationary point (the minimum) to a complex ('glassy') landscapes for $\mu < \mu_c$ with exponentially many stationary points, whose nature and statistics further depends on the properties of the covariance of the Gaussian function, see discussion below. Later on transitions of such type were suggested to be called the 'topology trivialization' transitions, see [21–24] and references therein. The mean number of stationary points was also studied recently for the case $N = 1$ [22] for the extended model of an elastic manifold of internal dimension d embedded in dimension N , where \mathbf{u} is generalized to $\mathbf{u}(x)$, with $x \in \mathbb{R}^d$. In that context the model (1) is also known as the $d = 0$ dimensional toy model of a particle in a random landscape. Note that it also arises in the study of the

Burgers equation with a random initial conditions in dimension N , which exhibits interesting transitions and regimes, see e.g. for $N = 1$ [30] and for large N [31].

The goal of the present paper is to address for the model (1) the density of eigenvalues of the $N \times N$ Hessian matrix $K_{ij}(\mathbf{u}_m)$, with \mathbf{u}_m being the point of the *global minimum* of the landscape: $\mathbf{u}_m = \text{Argmin}[\mathcal{H}(\mathbf{u})]$ and

$$K_{ij}(\mathbf{u}) = \frac{\partial^2}{\partial u_i \partial u_j} \mathcal{H}(\mathbf{u}) = \mu \delta_{ij} + \frac{\partial^2}{\partial u_i \partial u_j} V(\mathbf{u}) \quad (3)$$

being the landscape Hessian.

This distinguishes our paper from earlier studies of this or closely related models where the statistics is sampled over all saddle-points or minima at a given value of the potential $\mathcal{H}(\mathbf{u}) = E = \text{const}$, see e.g. [23, 28]. In doing this we combine the random matrix theory with the methods of statistical mechanics of disordered systems, as described in detailed below. In this way we show that depending on the type of the covariance structure $B(u)$ as classified below, the spectrum of the (positive semidefinite) Hessian at the global minimum may be either 'gapped' (i.e. position of the lowest spectral edge is strictly positive, being separated from zero in the limit $N \rightarrow \infty$ by a finite distance we call the 'gap', as it is conventional in statistical mechanics of disordered systems), or 'gapless', when the lowest eigenvalue stays exactly zero. More precisely, the landscapes with generically a single minimum are typical for parameter μ exceeding a critical value μ_c , and the Hessian at the global minimum of such simple landscape is always *gapped*. When approaching the transition point $\mu = \mu_c$ separating simple landscapes from "glassy" ones with exponentially many stationary points, the gap in the Hessian spectrum vanishes quadratically. For $\mu < \mu_c$ the Hessian spectrum is qualitatively different for 'moderately complex' and 'genuinely complex' landscapes. The 'moderately complex' landscapes are typical for short-range correlated random potentials and, in the statistical mechanics context, are closely related to the 1-step replica-symmetry breaking (1RSB) mechanism. Their Hessian spectra at the global minimum turn out to be again gapped, with the gap vanishing on approaching the transition point with a much higher exponent, see Fig. 1 below for a plot of the lower edge of the Hessian for one example of a short-ranged potential.

Let us now briefly outline a more quantitative description of our main results underlying the qualitative picture given above. We find that the mean (i.e. disorder-averaged) eigenvalue density $\rho(\lambda)$ for the Hessian at the global minimum is always given in the limit $N \rightarrow \infty$ by the semicircular law

$$\rho(\lambda) = 8 \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\pi(\lambda_+ - \lambda_-)^2}, \quad \int_{\lambda_-}^{\lambda_+} \rho(\lambda) d\lambda = 1 \quad (4)$$

the lower and upper spectral edges given by

$$\lambda_- = \mu_{eff} - 2\sqrt{B''(0)}, \quad \lambda_+ = \mu_{eff} + 2\sqrt{B''(0)} \quad (5)$$

and μ_{eff} very essentially depends on the type of the covariance in Eq.(2). For Short-Range correlated (SRC) potentials (represented, most importantly, by the family $B(q) = C/(q + \epsilon)^{\gamma-1}$, $\gamma > 1$, $C > 0$, $\epsilon > 0$) the parameter μ_{eff} can be represented as

$$\mu_{eff}^{(SRC)} = \mu + \frac{B''(0)}{\mu} + v(B'(\mathcal{Q}) - B'(0) - \mathcal{Q}B''(0)) \quad (6)$$

where the parameters $v > 0$, $\mathcal{Q} \geq 0$ satisfy the system of equations

$$\frac{\mu^2 \mathcal{Q}}{1 - \mu v \mathcal{Q}} = B'(\mathcal{Q}) - B'(0) \quad (7)$$

$$\frac{1}{v} \ln(1 - \mu v \mathcal{Q}) = -\mu \mathcal{Q} + v[B(\mathcal{Q}) - B(0) - \mathcal{Q}B'(\mathcal{Q})] \quad (8)$$

which together with Eq.(6) can be shown to imply the following representation for the lower spectral edge

$$\lambda_-^{(SRC)} = \left(\sqrt{\frac{\mu}{1 - \mu v \mathcal{Q}}} - \sqrt{\frac{1 - \mu v \mathcal{Q}}{\mu} B''(0)} \right)^2 \geq 0 \quad (9)$$

Note that the system of equations (7)-(8) is obviously always satisfied by $\mathcal{Q} = 0$ which defines the so called replica-symmetric solution. The solution is known to be correct (or 'stable') precisely in the phase with a single minimum: $\mu > \mu_c = \sqrt{B''(0)}$. In that case $\lambda_-^{(SRC)} = \frac{(\mu - \mu_c)^2}{\mu}$ and tends to zero quadratically on approaching μ_c from above. Below the transition for $\mu < \mu_c$ the system (7)-(8) allows a solution with $\mathcal{Q} > 0$ which defines the phase with 1-step broken replica symmetry. For a general SRC covariance $B(q)$ the system can be solved only numerically, see an example in Fig.1. Close to the critical value however, when $\delta = 1 - \frac{\mu}{\mu_c} \ll 1$ one can develop the perturbation theory in $0 < \delta \ll 1$. The calculation presented in this paper shows that the first nonvanishing contribution to $\lambda_-^{(SRC)}$ is order $O(\delta^4)$, and is given by

$$\lambda_-^{(SRC)} = \frac{\mu_c}{36[B'''(0)]^4} \left[\frac{3}{2}[B'''(0)]^2 - B''(0)B''''(0) \right]^2 \delta^4 + O(\delta^5) \quad (10)$$

In the Fig.2 we compare this asymptotics with the results of numerical solution of the system (7)-(8) for small δ and find the excellent agreement.

The coefficient in front of the leading term is generically strictly positive unless $\frac{3}{2}[B'''(0)]^2 - B''(0)B''''(0) = 0$. A more detailed study shows that the latter condition can be satisfied only for the logarithmically-correlated case with $B(0) - B(q) = C \ln(1 + \frac{q}{\epsilon})$; $C, \epsilon > 0$. Moreover, we will find that such vanishing is not a coincidence: for the logarithmic case the spectral gap λ_- is *identically zero* everywhere in the phase with broken replica symmetry, $\mu < \mu_c$, so that the associated spectrum is *gapless*. Moreover, the property turns out to be shared by all the 'genuinely complex' landscapes generated by the so-called Long-Range Correlated (LRC) potentials, most generically by those with the covariance $B(q) - B(0) =$

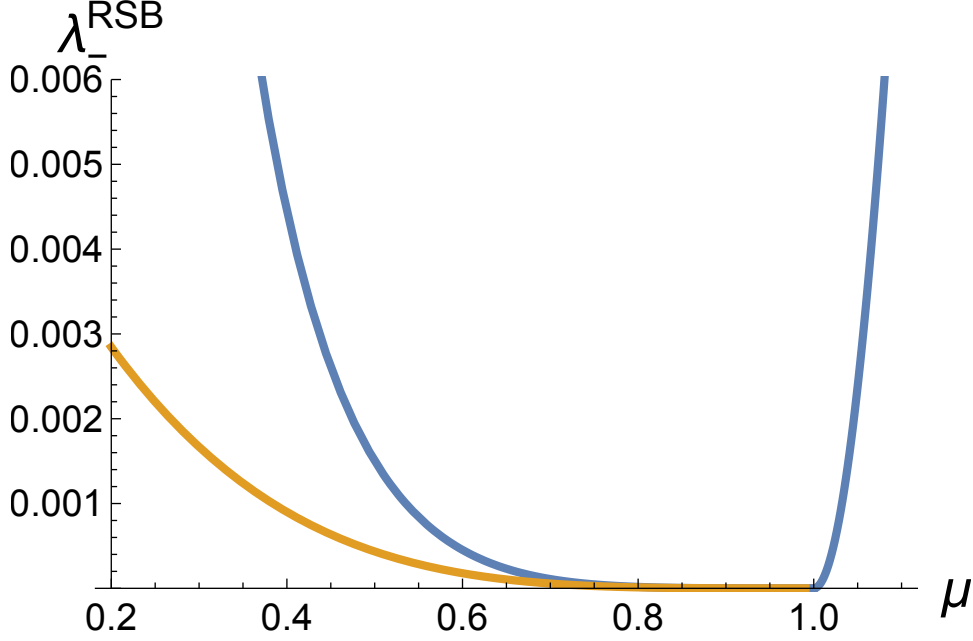


Figure 1. The lower edge of the Hessian spectrum $\lambda_-^{(\text{SRC})}$, associated to the 1 step replica symmetry breaking solution discussed in the text, plotted as a function of μ for the exponential SRC potential, with correlator $B(q) = e^{-q}$ (blue line for $\mu < 1$). It vanishes at the transition point for $\mu = \mu_c = 1$ as $\sim (\mu_c - \mu)^4$ (orange line: the asymptote given by formula (10)). Also plotted (blue line for $\mu > 1$) is the lower edge $\lambda_-^{(\text{SRC})} = (\mu - 1)^2/\mu$ for the replica symmetric solution above the transition point. The small μ behavior of $\lambda_-^{(1\text{RSB})}$ (not shown here) is discussed in the Appendix B.

$-C((q + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma})$, with $0 < \gamma < 1$ and $C, \epsilon > 0$. In the latter case however the parameter μ_{eff} is not given by Eq.(6) but by a more involved expression obtained in the framework of the Full Replica Symmetry Breaking scheme (FRSB) corresponding to a hierarchical picture of 'valleys within valleys within valleys' discovered in the context of spin glass models [33]. We show that in that case the Hessian at the global minimum remains gapless for all values of $\mu < \mu_c$.

Note that the properties of the Hessian at the global minimum determine the spectrum of the fluctuating modes in equilibrium, and is thus of fundamental importance. Clearly, the 'gapless' spectrum reflects an existence of very 'flat' directions in configuration space along which moving away from the global minimum incurs very little 'cost'. In the spin glass literature such flatness is known as a 'marginal stability' and is considered to be one of the fundamental properties of the systems with the hierarchical structure of the energy landscapes, see [32, 33] for early insights and [34] for a detailed more recent discussion of this and related questions. In particular, recently the 'marginal stability' has been shown to play the defining role in the spectrum of low-frequency modes of glasses [35]. In the latter work the spectrum of the soft modes was calculated in a mean-field model of the jamming transition, the so-called 'soft spherical perceptron'. The Hessian matrix in

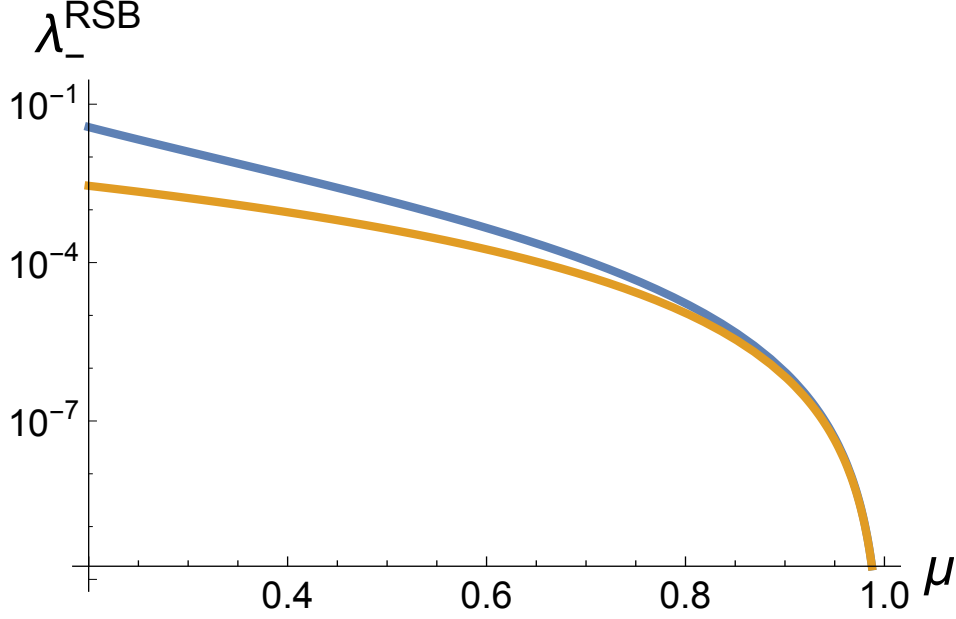


Figure 2. Semi-log plot of the lower edge of the Hessian spectrum $\lambda_-^{(\text{SRC})}$, in the 1 step replica symmetry breaking phase $\mu < 1$, plotted (in blue) as a function of μ for the same model as in Figure 1. It shows excellent agreement with the asymptotic analytical result (10) (orange line) near the transition point $\mu = \mu_c = 1$.

that model has the shape of a (uniformly shifted) Wishart matrix, whose spectrum is given by the (shifted) Marchenko-Pastur law. The model has two phases: 'RS simple' and 'FRSB complex' and the Marchenko-Pastur spectrum in that model was demonstrated to undergo a transition from gapped to gapless, similar to what we find here for Gaussian landscapes. Interestingly, in our model the Hessian spectrum is given by a shifted Wigner semicircle, rather than the Marchenko-Pastur law, with shifts and widths of the semicircle conspiring in an interesting way to produce the intricate picture just described. We also note that the excitation spectrum was shown to be gapless in a *quantum* spin glass model at zero temperature [36] (see also e.g. [37–40]). Marginally stable equilibria has been also shown to play an essential role in properties of large ecosystems [41].

Before describing our approach it is worth briefly mentioning an alternative model to Eq.(1) which is popular in studies of high-dimensional Gaussian landscapes. This is the so-called *spherical p-spin* model [42], which attracted a renewed attention recently [3,4,6,21,23,24]. In particular, early works [43] and [44] discussed Hessians for a variant of such model displaying transition from 'simple' to 'complex' landscapes of 1RSB type. Rather than concentrating on the global minimum, the works addressed the properties of the Hessians sampled over all stationary points at a given depth (called in [43] the 'energy density') e of the potential. In that context the authors found a threshold density e_{thres} such that Hessians for saddle points sampled at densities $e < e_{\text{thres}}$ are all positively gapped, whereas the gap tends to

zero when approaching $e = e_{thres}$ and becoming *negative* for $e > e_{thres}$. This picture suggested that all stationary points are minima for $e < e_{thres}$ but become saddle-points with unstable directions above the threshold. This conclusion was rigorously confirmed in [3, 4] and in the most recent studies [6, 23].

1.2. Statistical Mechanics Approach to the global minimum

In a useful interpretation, the model (1) represents an energy landscape of a single classical particle in high-dimensional random potential, and one is interested in building the associated statistical mechanics model. This can be done by introducing the canonical partition function \mathcal{Z}_β associated with the model at temperature T , with $\beta = 1/T > 0$ being the inverse temperature, as

$$\mathcal{Z}_\beta = \int_{\mathbb{R}^N} e^{-\beta \mathcal{H}(\mathbf{u})} d\mathbf{u}, \quad d\mathbf{u} = \prod_{i=1}^N du_i, \quad , \quad \beta = 1/T, \quad (11)$$

and considering the Boltzmann-Gibbs weights $\pi_\beta(\mathbf{u}) = \mathcal{Z}_\beta^{-1} e^{-\beta \mathcal{H}(\mathbf{u})}$ associated with any configuration \mathbf{u} . This interpretation allows to study properties of the landscape at the global minimum. One defines the thermal averaged value of any function $g(\mathbf{u})$ as $\langle g(\mathbf{u}) \rangle_T := \int g(\mathbf{u}) \pi_\beta(\mathbf{u}) d\mathbf{u}$. In the zero-temperature limit $\beta \rightarrow \infty$ the Boltzmann-Gibbs weights concentrate on the set of globally minimal values of the energy function $\mathcal{H}(\mathbf{u})$, so that for any well-behaving function $\langle g(\mathbf{u}) \rangle_T$ should tend to the value of $g(\mathbf{u}_m)$ evaluated at the argument $\mathbf{u}_m = \text{Argmin}[\mathcal{H}(\mathbf{u})]$. Although this fact is valid in every disorder realization, in practice one concentrates on finding the disorder-averaged values $\overline{\langle g(\mathbf{u}) \rangle}_T$. In particular, in this paper we choose the function $g(\mathbf{u})$ as the resolvent of the Hessian:

$$g(\mathbf{u}) := G(\lambda; \mathbf{u}) = \frac{1}{N} \text{Tr} \frac{1}{\lambda - K(\mathbf{u})}, \quad \lambda \in \mathbb{C} \quad (12)$$

From this we proceed to calculating the mean spectral density of the Hessian eigenvalues “at a temperature T ”, defined as

$$\rho_T(\lambda) = \frac{1}{\pi} \lim_{\text{Im } \lambda \rightarrow 0^-} \text{Im} \overline{\langle G(\lambda, \mathbf{u}) \rangle}_T \quad (13)$$

our final aim being to obtain the mean spectral density of Hessian eigenvalues at the absolute minimum by setting temperature to zero:

$$\rho(\lambda) = \lim_{T \rightarrow 0} \rho_T(\lambda) = \frac{1}{\pi} \lim_{\text{Im } \lambda \rightarrow 0^-} \text{Im} \overline{G(\lambda, \mathbf{u}_m)} \quad (14)$$

For our goals we find it most technically transparent to represent the resolvent using the formal *first* replica limit identity:

$$G(\lambda, \mathbf{u}) := \frac{1}{N} \text{Tr} \frac{1}{\lambda - K(\mathbf{u})} = -2 \lim_{m \rightarrow 0} \frac{1}{mN} \partial_\lambda [\det(\lambda - K(\mathbf{u}))]^{-\frac{m}{2}} \quad (15)$$

and use a representation of the inverse determinants in terms of m replicated Gaussian integrals over N -component real-valued vectors ϕ_α , with $\alpha = 1, \dots, m$:

$$G(\lambda, \mathbf{u}) = -2 \lim_{m \rightarrow 0} \frac{1}{mN} \partial_\lambda \int_{\mathbb{R}^{Nm}} \prod_{\alpha=1}^m d\phi_\alpha e^{-\frac{i}{2}\lambda \sum_{\alpha=1}^m \phi_\alpha^2} e^{\frac{i}{2} \sum_{\alpha=1}^m \phi_\alpha \cdot K(\mathbf{u}) \cdot \phi_\alpha} \quad (16)$$

where we set the factor $(\frac{i}{\pi})^{m/2} \rightarrow 1$ for $m = 0$. The problem therefore amounts to first calculating the disorder and thermal averaged value

$$\overline{\langle G(\lambda, \mathbf{u}) \rangle_T} = -2 \lim_{m \rightarrow 0} \frac{1}{mN} \partial_\lambda \int_{\mathbb{R}^{Nm}} \prod_{\alpha=1}^m d\phi_\alpha e^{-\frac{i}{2}\lambda \sum_{\alpha=1}^m \phi_\alpha^2} \overline{\langle e^{\frac{i}{2} \sum_{\alpha=1}^m \phi_\alpha \cdot K(\mathbf{u}) \cdot \phi_\alpha} \rangle_T} \quad (17)$$

where

$$\langle e^{\frac{i}{2} \sum_{\alpha=1}^m \phi_\alpha \cdot K(\mathbf{u}) \cdot \phi_\alpha} \rangle_T = \mathcal{Z}_\beta^{-1} \int_{\mathbb{R}^N} d\mathbf{u} e^{\frac{i}{2} \sum_{\alpha=1}^m \phi_\alpha \cdot K(\mathbf{u}) \cdot \phi_\alpha - \beta \mathcal{H}(\mathbf{u})} \quad (18)$$

and then, by performing the zero-temperature limit, to capture the contribution from the global minimum only.

The task of evaluating the expectation over the random variables contained in the random 'energy function' $\mathcal{H}(\mathbf{u})$ in both the numerator and the denominator of the Gibbs measure in Eq.(18) is one of the central problems in the theory of disordered systems. Although the problem is meaningful and interesting for any N , we will be mainly concerned with the limit of large $N \gg 1$, where we will develop a systematic method of analysis. For the model in question the associated methods are known, and we give the necessary background in the next two sections.

1.3. Two classes of random potentials

Before recalling the known results on the statistical mechanics of the model for large N , let us first specify the types of random potential functions $V(\mathbf{u})$, defined for arbitrary N , that we study here. An important characteristics is the behavior of the correlator $B(q)$ for large q , and we will generally define γ such that

$$B(q) \sim q^{1-\gamma} \quad (19)$$

as in [45, 46] (although more general choices of functions are possible, as we now discuss).

According to Yaglom [29], there are two essentially different types of Gaussian random potentials whose covariance function $B(q)$, defined in (2), belongs to the class \mathcal{D}_∞ .

- The first type corresponds to genuine isotropic random potentials, and for those the covariances $B(q)$ are characterized by a non-negative normalizable "spectral density function" $\tilde{B}(k) \geq 0$, $k \geq 0$ in terms of which $B(q)$ is represented as (see [29] p.354)

$$B(q) = \int_0^\infty e^{-k^2 q} \tilde{B}(k) dk, \quad B(0) = \int_0^\infty \tilde{B}(k) dk < \infty. \quad (20)$$

In particular, $B(q)$ is decreasing and convex, i.e. satisfies $B'(q) < 0, B''(q) > 0 \forall q \geq 0$, and in addition $B'(q \rightarrow \infty) = 0$. Here and below the number of dashes indicates the order of derivatives taken. In the physics literature such a class of random potentials is often called "short range correlated" (SRC). The most important example in this class is the power-law correlated potential with $B(q) = C/(q + \epsilon)^{\gamma-1}$, $\gamma > 1$, $C > 0$, which for $\epsilon > 0$ corresponds to smooth potentials. There are other interesting families listed in [29], the majority of which however produces very rough realizations of potentials. If however one insists (as is needed for our goals here, see below) on the finiteness of at least two derivatives of $B(q)$ at $q = 0$, then deserves mentioning also the case $B(q) = Ce^{-aq}$, $C > 0$, (which can be considered as a limiting case $\gamma \rightarrow \infty$ of the power-law correlated potentials), as well as the so-called Matern family $B(q) = C(aq)^{\nu/2} K_{\nu}(\sqrt{aq})$ with $C > 0$, $a > 0$ and 'roughness parameter' ν , where $K_{\nu}(x)$ stands for the modified Bessel (a.k.a. Macdonald) function and the condition $\nu > k$ ensures that the process is at least k times differentiable.

- The second type of covariances occurs in the situation when the normalization integral $\int_0^{\infty} \tilde{B}(k) dk$ diverges. It corresponds to so-called "long range correlated" (LRC) random potentials with isotropic *increments*, also known as *locally isotropic random fields*, see e.g. [29], p.438. The spectral density function $\tilde{B}(k)$ is again non-negative, and must now satisfy the condition $\int_0^{\infty} \frac{k^2}{k^2+1} \tilde{B}(k) dk < \infty$. Such conditions allow one to prove that in any dimension $N \geq 1$ there exists a random potential whose *structure function* $\frac{1}{2}(V(0) - V(\mathbf{u}))^2 = B(0) - B(q)$, $q = u^2/(2N)$, is well defined and given by

$$B(0) - B(q) = \int_0^{\infty} dk (1 - e^{-k^2 q}) \tilde{B}(k) + Au, \quad A \geq 0. \quad (21)$$

In what follows we will impose an additional requirement $B'(q \rightarrow \infty) = 0$, which ensures $A = 0$. The most widely-known example of the locally isotropic LRC random potential is the so-called *self-similar* random potential, see [29] p. 441, characterized by the spectral density $\tilde{B}(k > 0) = k^{2\gamma-3}$, $0 < \gamma < 1$. The corresponding covariance behaves as

$$B(q) - B(0) = C_{\gamma} q^{1-\gamma}. \quad (22)$$

which is the well-known example of the high-dimensional analogue of the fractional Brownian motion (and reduces to the Brownian motion for $N = 1$ and $\gamma = 1/2$). As such it is not smooth, but can be made smooth by introducing a proper cutoff for $\tilde{B}(k)$ at large \tilde{k} , without changing the behavior (22) at large q (see below).

It is important to note that for the Hessian entries $K_{ij} = \mu \delta_{ij} + \frac{\partial^2}{\partial u_i \partial u_j} V(\mathbf{u})$ to be well defined, the potential should be twice differentiable, hence $B''(0)$ should exist. This is what we will assume here, for both classes of potentials. It corresponds

to random potentials usually considered in the theory of pinning, for which the existence of a so-called Larkin regime, and Larkin scale, is related to $B''(0)$ being finite (see [45–50] and references therein). In fact μ_c defined below is related to $B''(0)$, and defines the Larkin mass of the problem. On the other hand, what matters for the physical applications (and defines the nature of the low-temperature phase in the present model) is the *large-distance* behaviour of the random potential, controlled by $B(q \gg 1)$. Hence we will always assume some small-scale cutoff parameter $\epsilon > 0$, see examples below, to remain in that class of sufficiently smooth potentials. This is easily realizable in the class \mathcal{D}_∞ by choosing $\tilde{B}(k)$ fast enough decaying at large k with their four lowest moments defined.

1.4. Statistical Mechanics of disordered high-dimensional landscapes

Let us now discuss the statistical mechanics of the model in the large N limit. Such simple yet non-trivial models play the role of a laboratory for developing the methods to deal with problems of statistical mechanics where an interplay between thermal fluctuations and quenched disorder is essential. Paradigmatic examples are spin glasses [33], but similar effects occur for polymers in random environments, for phase separating interfaces in random field models or for elastic manifolds pinned by random impurities. In general, the presence of quenched disorder leaves little choice but to employ the so-called replica trick, which is a powerful heuristic way of extracting the averaged free energy of the system from the moments of the partition function. Note that considerable progress has been achieved in the last decade in developing the rigorous aspects of that theory [51, 52]. For the present model, a detailed rigorous study was done by Klimovsky [53], so one may hope that the main results of the present paper can be eventually made rigorous as well.

Technically, the problem amounts to calculating the ensemble average of the equilibrium free energy $F = -T \ln \mathcal{Z}$. The replica trick allows one to represent integer moments $\overline{\mathcal{Z}^n}$ of the partition function in a form of some multivariable non-Gaussian integrals, and one then faces the usual problem of finding ways to evaluate these integrals explicitly enough to be able to perform the replica limit $n \rightarrow 0$. In systems of high dimension this can be done using saddle point methods, which allow to employ the intricate pattern of spontaneous replica symmetry breaking (1RSB or FRSB) discovered by Parisi [54] in the framework of the Sherrington-Kirkpatrick model of spin glass with infinite-range interactions.

In this broad context, the model Eq (1) has a long history. First, in order to have *bona fide* thermodynamic properties, the partition function \mathcal{Z}_β of the model should be well-defined for any realization of the random potential. This is achieved by the confining non-random part $V_{con}(\mathbf{x}) = \frac{\mu}{2} \mathbf{x}^2$ which prohibits escape of the particle to infinity. The curvature $\mu > 0$ then plays, together with the temperature T , the role of the main control parameter, and one of standard goals of the theory is to investigate the phase diagram in the (μ, T) plane. Studies of the phase diagram in the limit of

large N , and of detailed physical properties, were performed in [45, 46, 55, 56] for the statics, and in [57, 58] for the aging dynamics (which does present strong relations with the statics). More general classes of confining potentials were considered in [19]. The generalized model of elastic manifolds, which reduces to Eq (1) for $d = 0$, exhibits quite similar features and was also much studied [45, 59], especially in the context of pinned vortex lattices [47–49] and in comparison to the functional RG theory of pinning [50, 60].

From all these studies it is known that the nature of the low-temperature phase of the model crucially depends on the behaviour of the covariance function $B(q)$ for large arguments, i.e. the exponent γ in (19) (see below for a more precise statement), leading to the distinction ‘short-ranged’ correlated (SRC) or ‘long-ranged’ (LRC). It turns out that for the model Eq (1) this distinction coincides with the two classes discussed in the previous section (note that for the more general model of a manifold, whose Hessian will be studied in a separate paper [61], there is a critical value such that LRC holds for $\gamma < \gamma_c(d)$, with $\gamma_c(0) = 1$). We will thus use this denomination everywhere. In the statistical mechanics of the model the SRC potentials correspond to a 1 step replica symmetry breaking (1RSB) low temperature phase, while the LRC potentials lead to the full RSB pattern (FRSB). The precise criterion to classify covariance structures at large N , as SRC vs LRC, was studied in much detail in [56] for the model Eq (1) (see also [58]), and for the more general manifold problem in [50] (see also [45]), two works to which we refer extensively below.

The distinction between the two cases has been put in a nice compact form in [56] (see also ‡). It employs the notion of the so-called Schwarzian derivative $\{B'(q), q\} = -A(q)/[B''(q)]^2$, where $A(q)$ is expressed in terms of $B(q)$ as

$$A(q) = \frac{3}{2} [B'''(q)]^2 - B''(q)B''''(q). \quad (23)$$

In terms of $A(q)$ it was demonstrated that

- any potential whose covariance function satisfies the condition $A(q) > 0 \forall q \geq 0$ must have the 1RSB low temperature phase. It is easy to check that such a situation includes, in particular, all standard families of the SR potentials with covariances listed after Eq.(20). It is natural to conjecture that this property is characteristic for all SR isotropic potentials with covariances defined via Eq.(20).
- Any potential whose correlation function satisfies the condition $A(q) < 0 \forall q \geq 0$ must necessarily exhibit the FRSB low temperature phase. This condition holds for the standard LR correlation functions of the type Eq.(21), i.e. for $B(q) - B(0) = -g^2((q + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma})$, with $0 < \gamma < 1$, $\epsilon > 0$, obtained from the choice $\tilde{B}(k) = \frac{2g^2}{\Gamma(\gamma-1)} k^{2\gamma-3} e^{-k^2\epsilon}$.

‡ This classification is also obtained, more generally for manifolds, by requesting a monotonous order parameter from the full RSB solution, see e.g. Eqs 8.16 in [60].

It is natural to conjecture that typical smooth LR random potentials with independent increments should be of this type.

Clearly, the above criterion naturally singles out as a special marginal case random potentials satisfying $A(q) = 0$. The only function satisfying this condition globally, i.e for all $q \geq 0$, and satisfying also the requirement $B'(q \rightarrow \infty) = 0$ is indeed given by a *logarithmic* correlation function, of the type studied in [63]:

$$B(0) - B(q) = g^2 \ln \left(1 + \frac{q}{\epsilon} \right), \quad (24)$$

where g and $\epsilon > 0$ are given constants. Let us stress that the expression Eq.(24) is a legitimate covariance function belonging to the \mathcal{D}_∞ class of LR locally isotropic fields, Eq.(21). Indeed, it can be considered as a limiting case $\gamma \rightarrow 1$ of the smooth LCR potential discussed above and corresponds to the spectral density of the form $\tilde{B}(k) = \frac{2g^2}{k} e^{-\epsilon k^2}$, which satisfies the required condition $\int_0^\infty \frac{k^2}{k^2+1} \tilde{B}(k) dk < \infty$.

As shown in several works [47–49, 56, 58] such logarithmically-correlated potentials induce thermodynamics with features of both the SR-1RSB and LR-FRSB regimes, and are singled out in many respects. In particular, one can show that this case is the only § infinite-dimensional model where the transition temperature remains finite when the confining parameter $\mu \rightarrow 0$. In fact, such a case can be considered as an (infinitely) high-dimensional representative of an intensively studied class of log-correlated random processes and fields in *low dimensions* $N = 1, 2$. These processes, characterized by logarithmic singularity on the diagonal of its covariance kernel [63], appear with surprising regularity in many different contexts, such as the statistical mechanics of branching random walks and polymers on trees [64, 65], and associated extremal value problems [63, 66–70] of great relevance from random matrix theory and number theory [71, 72], to probabilistic description of two-dimensional gravity [73, 74]. In fact, many generic features of the low dimensional log-correlated potential are naturally expected to be shared by its high-dimensional counterpart, the latter deserves to be studied seriously in every aspect.

In the next section we perform the analysis of Eqs.(17) and (18) by employing the *second* replica trick, adapting the considerations in works [50, 56] to the Hessian problem, with due modifications.

2. Replica analysis of the Hessian problem

2.1. Main idea of the calculation

In the framework of the replica trick we represent the normalization factor \mathcal{Z}_β^{-1} in Eq.(18) formally as $1/\mathcal{Z}_\beta = \lim_{n \rightarrow 0} \mathcal{Z}_\beta^{n-1}$ and treat the parameter n before the limit as a positive integer. After this is done, averaging the product of n integrals over the Gaussian potential $V(\mathbf{u})$ is an easy task. Before embarking on the detailed

§ One may also consider a more general class of *multiscale* log-correlated fields sharing this feature,

calculation let us sketch the main idea. We will rewrite the disorder average of Eq.(18)

$$\overline{\langle e^{\frac{i}{2} \sum_{\alpha=1}^m \phi_{\alpha} \cdot K(\mathbf{u}) \cdot \phi_{\alpha}} \rangle_T} = \lim_{n \rightarrow 0} \int \prod_{a=1}^n d\mathbf{u}_a e^{-L_{n,m}[\mathbf{u}, \phi]} \quad (25)$$

in terms of some replicated action $L_{n,m}[\mathbf{u}, \phi]$ that we will calculate. This allows to write (17) as

$$\overline{\langle G(\lambda, \mathbf{u}) \rangle_T} = \lim_{m \rightarrow 0, n \rightarrow 0} \int_{\mathbb{R}^{N(n+m)}} \prod_{\alpha=1}^m d\phi_{\alpha} \prod_{a=1}^n d\mathbf{u}_a \frac{i \sum_{\alpha=1}^m \phi_{\alpha}^2}{mN} e^{-\frac{i}{2} \lambda \sum_{\alpha=1}^m \phi_{\alpha}^2} e^{-L_{n,m}[\mathbf{u}, \phi]} \quad (26)$$

where we have performed the derivative w.r.t. λ . Note that formally setting $\frac{i \sum_{\alpha=1}^m \phi_{\alpha}^2}{mN} \rightarrow 1$ in (26) is equivalent to setting $\frac{-2}{mN} \partial_{\lambda} = 1$ in the r.h.s of (15), which gives unity. Hence the integral in (26) is normalized to unity for $n = m = 0$, and we can rewrite (26) as an expectation value in the replicated theory

$$\overline{\langle G(\lambda, \mathbf{u}) \rangle_T} = \lim_{m \rightarrow 0, n \rightarrow 0} \left\langle \frac{i \sum_{\alpha=1}^m \phi_{\alpha}^2}{mN} \right\rangle_{n,m} \quad (27)$$

where $\langle \mathcal{O} \rangle_{n,m}$ denotes the expectation value of \mathcal{O} in the theory with $n \times m$ replicated fields $\mathbf{u}_a, \phi_{\alpha}$ and the action $L_{n,m}[\mathbf{u}, \phi]$. Below we perform the calculation of this expectation value at large N using the saddle point method.

2.2. Calculation

We have at the first step, from (18) and (3):

$$\begin{aligned} \overline{\langle e^{\frac{i}{2} \sum_{\alpha=1}^m \phi_{\alpha} \cdot K(\mathbf{u}) \cdot \phi_{\alpha}} \rangle_T} &= \lim_{n \rightarrow 0} e^{\frac{i\mu}{2} \sum_{\alpha=1}^m \phi_{\alpha}^2} \int_{\mathbb{R}^{Nn}} \prod_{a=1}^n d\mathbf{u}_a e^{-\frac{\beta\mu}{2} \sum_{a=1}^n \mathbf{u}_a^2} \\ &\quad \times e^{\frac{1}{2} \left[-\beta \sum_{a=1}^n V(\mathbf{u}_a) + \frac{i}{2} \sum_{ij=1}^N \sum_{\alpha=1}^m \phi_{\alpha}^i \phi_{\alpha}^j \frac{\partial^2}{\partial u_{1i} \partial u_{1j}} V(\mathbf{u}_1) \right]^2} \end{aligned} \quad (28)$$

Now, by differentiating in Eq.(2) one find the following relations:

$$\left\langle V(\mathbf{u}) \frac{\partial^2}{\partial u_i \partial u_j} V(\mathbf{u}) \right\rangle = \delta_{ij} B'(0) \quad (29)$$

$$\left\langle \frac{\partial^2}{\partial u_i \partial u_j} V(\mathbf{u}) \frac{\partial^2}{\partial u_k \partial u_l} V(\mathbf{u}) \right\rangle = \frac{1}{N} B''(0) (\delta_{ij} \delta_{lk} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (30)$$

and for $a \neq 1$:

$$\begin{aligned} \left\langle V(\mathbf{u}_a) \frac{\partial^2}{\partial u_{1i} \partial u_{1j}} V(\mathbf{u}_1) \right\rangle &= \delta_{ij} B' \left(\frac{(\mathbf{u}_a - \mathbf{u}_1)^2}{2N} \right) \\ &\quad + \frac{(\mathbf{u}_{ai} - \mathbf{u}_{1i})(\mathbf{u}_{aj} - \mathbf{u}_{1j})}{N} B'' \left(\frac{(\mathbf{u}_a - \mathbf{u}_1)^2}{2N} \right) \end{aligned} \quad (31)$$

which, brings Eq.(28) to the form Eq.(25) with

$$L_{n,m}[\mathbf{u}, \boldsymbol{\phi}] = \mathcal{L}_m[\boldsymbol{\phi}] + \mathcal{L}_{n,m}[\mathbf{u}, \boldsymbol{\phi}] \quad (32)$$

where the \mathbf{u} -independent part of the action is given by

$$\mathcal{L}_m[\boldsymbol{\phi}] = \frac{B''(0)}{8N} \left[\left(\sum_{\alpha=1}^m \phi_\alpha^2 \right)^2 + 2 \sum_{\alpha, \beta=1}^m (\phi_\alpha \cdot \phi_\beta)^2 \right] - \frac{i\mu}{2} \sum_{\alpha} \phi_\alpha^2 \quad (33)$$

whereas both \mathbf{u} - and $\boldsymbol{\phi}$ - dependent part is

$$\begin{aligned} \mathcal{L}_{n,m}[\mathbf{u}, \boldsymbol{\phi}] &= \frac{\beta\mu}{2} \sum_{a=1}^n \mathbf{u}_a^2 - N \frac{\beta^2}{2} \sum_{a,b=1}^n B \left(\frac{(\mathbf{u}_a - \mathbf{u}_b)^2}{2N} \right) \\ &+ \frac{i\beta}{2} \left(\sum_{\alpha} \phi_\alpha^2 \right) \sum_{a=1}^n B' \left(\frac{(\mathbf{u}_a - \mathbf{u}_1)^2}{2N} \right) + \frac{i\beta}{2N} \sum_{a=1}^n B'' \left(\frac{(\mathbf{u}_a - \mathbf{u}_1)^2}{2N} \right) \sum_{\alpha=1}^m ((\mathbf{u}_a - \mathbf{u}_1) \cdot \boldsymbol{\phi}_\alpha)^2 \end{aligned} \quad (34)$$

Let us now examine the expression (26) for $\overline{\langle G(\lambda, \mathbf{u}) \rangle}_T$ which has the form of an integral over $\mathbb{R}^{N(n+m)}$ for the $n+m$ N -component vectors \mathbf{u}_a and $\boldsymbol{\phi}_\alpha$. We see that, given the expression (32),(33),(34) for the replicated action $L_{n,m}[\mathbf{u}, \boldsymbol{\phi}]$ the integrand in (26) depends only on the vectors \mathbf{u}_a and $\boldsymbol{\phi}_\alpha$ through the combinations

$$Q_{ab} = \mathbf{u}_a \cdot \mathbf{u}_b \quad , \quad P_{\alpha\beta} = \boldsymbol{\phi}_\alpha \cdot \boldsymbol{\phi}_\beta \quad , \quad R_{a\alpha} = \mathbf{u}_a \cdot \boldsymbol{\phi}_\alpha \quad (35)$$

We can embody these scalar products into a $(n+m) \times (n+m)$ matrix M of scalar products of combined vectors $(\mathbf{u}_a, \boldsymbol{\phi}_\alpha)$ as

$$M = \begin{pmatrix} Q & R \\ R^T & P \end{pmatrix} \geq 0, \quad (36)$$

We can thus trade the integration variables \mathbf{u}_a and $\boldsymbol{\phi}_\alpha$ for the matrix M by using the invariant integration theorem, recalled in the Appendix A, upon the change $n \rightarrow n+m$, $\mathbf{u}_a \rightarrow (\mathbf{u}_a, \boldsymbol{\phi}_\alpha)$ and $Q \rightarrow M$ of the formula there. We also perform a final rescaling $Q, R, P \rightarrow NQ, NR, NP$ and obtain

$$\overline{\langle G(\lambda, \mathbf{u}) \rangle}_T = \lim_{m \rightarrow 0, n \rightarrow 0} \mathcal{I} \left[\frac{i \operatorname{tr} P}{m} \right] \quad (37)$$

where we denote the integral

$$\mathcal{I}[\mathcal{O}] = \mathcal{C}_{N,n+m}^{(o)} N^{\frac{n+m}{2}N} \int_{M \geq 0} dM \mathcal{O}(M) e^{-NL(M)} (\det M)^{-\frac{n+m+1}{2}} \quad (38)$$

of any observable $\mathcal{O}(M)$, which is now over the space of symmetric real positive semi-definite matrix M with measure $dM = dQdPdR$ where

$$dQ = \prod_{1 \leq a \leq b \leq n} dQ_{ab} \quad , \quad dP = \prod_{1 \leq \alpha \leq \beta \leq m} dP_{\alpha\beta} \quad , \quad dR = \prod_{a=1}^n \prod_{\alpha=1}^m dR_{a\alpha} \quad (39)$$

Note that Q and P are also symmetric real positive semi-definite matrices (we can consider them in fact positive definite as the manifold of exactly zero eigenvalues forms the boundary of the integration domain and is therefore of lower dimension. Such boundaries can not contribute to the integral unless the integrand is manifestly divergent when approaching the boundary manifold. One can check *a posteriori* this is not the case in our problem). The off-diagonal block R is a general real matrix such that M remains positive definite. The latter condition imposing certain restrictions on the maximal singular value of R (see about the singular values below) is however irrelevant for our consideration below. The prefactor is given by (see the Appendix A)

$$\mathcal{C}_{N,n+m}^{(o)} = \frac{\pi^{\frac{n+m}{2} \left(N - \frac{n+m-1}{2}\right)}}{\prod_{k=0}^{n+m-1} \Gamma\left(\frac{N-k}{2}\right)} \quad (40)$$

We have also defined the action

$$L[M] = L_m[P] + L_{n,m}[P, Q, R] \quad (41)$$

$$L_m[P] = \frac{B''(0)}{8} [(\text{tr } P)^2 + 2 \text{tr } (P^2)] + \frac{i(\lambda - \mu)}{2} \text{tr } P \quad (42)$$

$$L_{n,m}[P, Q, R] = -\frac{1}{2} \text{tr } \log M + \frac{\beta\mu}{2} \text{tr } Q - \frac{\beta^2}{2} \sum_{a,b=1}^n B \left(\frac{Q_{aa} + Q_{bb} - 2Q_{ab}}{2} \right) \quad (43)$$

$$\begin{aligned} & + \frac{i\beta}{2} \text{tr } P \sum_{a=1}^n B' \left(\frac{Q_{aa} + Q_{a1} - 2Q_{11}}{2} \right) \\ & + \frac{i\beta}{2} \sum_{a=1}^n B'' \left(\frac{Q_{aa} + Q_{a1} - 2Q_{11}}{2} \right) ((RR^T)_{aa} + (RR^T)_{11} - 2(RR^T)_{a1}) \end{aligned} \quad (44)$$

The above expression is exact for any $N > n + m$.

Note again that, by the same argument as below Eq. (26) applied to (37) shows that $\lim_{m \rightarrow 0, n \rightarrow 0} \mathcal{I}[1] = 1$. Hence we can write again the mean resolvent as an expectation value

$$\overline{\langle G(\lambda, \mathbf{u}) \rangle_T} = \lim_{m \rightarrow 0, n \rightarrow 0} \left\langle \frac{i \text{tr } P}{m} \right\rangle_L \quad (45)$$

w.r.t. the action $L(M)$, where the expectation value of \mathcal{O} is defined as $\langle \mathcal{O} \rangle_L = \mathcal{I}[\mathcal{O}] / \mathcal{I}[1]$.

Finally, note that the integrand in Eqs.(37-38), including the action $L(M)$, is invariant by the transformation

$$P \rightarrow \mathcal{O}^T P \mathcal{O} \quad , R \rightarrow R \mathcal{O} \quad (46)$$

for any orthogonal matrix \mathcal{O} in $O(m)$ (a consequence of the original invariance upon rotating the m replica ϕ_α).

2.3. Saddle point solution and the semi-circle density of Hessian eigenvalues

The form of the integrand in Eqs.(37) is suggestive of evaluating the corresponding P -, Q - and R integrals by the saddle-point method for $N \gg 1$, assuming, as usual, commutativity of the replica limit(s) with $N \rightarrow \infty$. Hence we will now search for the stationary points, M^{sp} , of $L(M)$. To leading order as $N \rightarrow +\infty$ we will replace Eq.(45) by its value at the stationary points

$$\overline{\langle G(\lambda, \mathbf{u}) \rangle_T} = \lim_{m \rightarrow 0, n \rightarrow 0} \frac{i \operatorname{tr} P^{\text{sp}}}{m} \quad (47)$$

where P^{sp} is the P - component of the matrix M^{sp} (note that because of symmetries such as (46), there is a manifold of stationary points, however the value of $\operatorname{tr} P^{\text{sp}}$ is the same for all these stationary points). Since Eq.(45) is an expectation value, we do not need to calculate explicitly below all prefactors in the saddle point evaluation, and only the value of $\operatorname{tr} P$ on the saddle point manifold matters to the leading order at large N (the prefactors would matter if one aimed to carry out a $1/N$ expansion).

Although $L(M)$ looks complicated, the first simplifying feature is that one can argue that the saddle-point solution corresponds to the choice $R = 0$. Indeed, the saddle point equation, expressed along the R components of M , $\frac{\delta}{\delta R} L = 0$, reads, for $a = 1, \dots, n$, $\alpha = 1, \dots, m$

$$\begin{aligned} (M^{-1})_{a\alpha} = i\beta \left[B'' \left(\frac{Q_{aa} + Q_{a1} - 2Q_{11}}{2} \right) (R_{a\alpha} - R_{1\alpha}) \right. \\ \left. + \delta_{a1} \sum_{b=1}^n B'' \left(\frac{Q_{bb} + Q_{b1} - 2Q_{11}}{2} \right) (R_{1\alpha} - R_{b\alpha}) \right] \end{aligned} \quad (48)$$

and since the inversion of Eq.(36) leads to

$$M^{-1} = \begin{pmatrix} \tilde{Q} & \tilde{R} \\ \tilde{R}^T & \tilde{P} \end{pmatrix}, \quad \tilde{R} = -Q^{-1}R [P - R^T Q^{-1}R]^{-1} \quad (49)$$

we see that $(M^{-1})_{a\alpha} \equiv \tilde{R}_{a\alpha}$ vanishes as long as $R \rightarrow 0$, hence the choice $R = 0$ always satisfies the saddle-point equation Eq.(48). To show that the solution $R = 0$ is the relevant one, one needs to study the stability of the quadratic form by expanding to second order in R : We have

$$-\frac{1}{2} \operatorname{tr} \log M = -\frac{1}{2} \operatorname{tr} \log \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} - \frac{1}{2} \operatorname{tr} \log \left(I + \begin{pmatrix} 0 & RP^{-1} \\ R^T Q^{-1} & 0 \end{pmatrix} \right) \quad (50)$$

$$= -\frac{1}{2} \operatorname{tr} \log Q - \frac{1}{2} \operatorname{tr} \log P + \frac{1}{2} \operatorname{tr} RP^{-1}R^T Q^{-1} + O(R^4) \quad (51)$$

Hence this part of the action starts at the terms of second order in entries of R around $R = 0$, and the stability matrix for M around the $R = 0$ saddle point decouples (i.e. there are no quadratic terms of the kind QR or PR). To check

stability of $R = 0$ solution one therefore only needs to check that the second-order terms in Eq.(51) can be represented as a positive definite quadratic form in the components of R (note that the only other R -dependent term in $L(M)$, i.e. the last term of Eq.(44), being purely imaginary cannot modify the stability properties with respect to fluctuations in R).

In checking this consider first a special case $m = n$ such that R is a square matrix. Any such R can be represented as $R = \mathcal{O}_1 R_d \mathcal{O}_2^T$, where $\mathcal{O}_1 \in O(n)$ and $\mathcal{O}_2 \in O(n)$ are two different orthogonal matrices, diagonalizing RR^T and $R^T R$, correspondingly, and $R_d = \text{diag}(r_1, r_2, \dots, r_n)$, with $r_a \geq 0$, $a = 1, \dots, n$ being the so-called singular values of R . Defining $\mathcal{O}_1^T Q^{-1} \mathcal{O}_1 = A > 0$ and $B = \mathcal{O}_2^T P^{-1} \mathcal{O}_2 > 0$ we have

$$\text{tr } R^T Q^{-1} R P^{-1} = \text{tr } R_d A B R_d = \sum_{ab} r_a r_b C_{ab},$$

where $C_{ab} = A_{ab} B_{ab}$ (we used $B^T = B$). Matrices C of this form are known as the 'Hadamard product' of A and B , and according to the 'Schur product theorem' one has $C > 0$ as long as $A > 0$ and $B > 0$, showing that the associated quadratic form in singular values is positive definite. As the result the value dominating the R -integration always corresponds to $R_d = 0$, hence to the whole matrix $R = 0$. The same consideration can be straightforwardly extended to $n \neq m$ case, via representing the $m \times n$ matrix R via a very similar singular value decomposition with $\mathcal{O}_2 \in O(m)$ and R_d being $m \times n$ diagonal matrix which is zero at the saddle-point. Thus, from now on we may assume R to be integrated out in the quadratic approximation, i.e. we set $R = 0$ in the saddle point and deal with only the diagonal blocks Q and P (as noted above we do not need, to the leading order at large N , to keep track of the pre-exponential factors produced by the R integration).

It is now useful to observe that the form of the P -dependence in the integrand Eqs.(37)-(38), invariant under Eq.(46), suggests diagonalizing P by orthogonal rotations as $P = O_P P_d O_P^{-1}$, $P_d = \text{diag}(p_1, \dots, p_m) > 0$ and integrating out the associated orthogonal matrices (which incurs the change of measure $dP \rightarrow dP_d \prod_{\alpha < \beta} |p_\alpha - p_\beta|$). We can thus restrict our analysis to finding the stationary point of

$$L(M)|_{R=0} = \phi_n(Q) + \sum_{\alpha=1}^m \mathcal{L}_Q(p_\alpha) + \frac{B''(0)}{8} \left(\sum_{\alpha=1}^m p_\alpha \right)^2 \quad (52)$$

where we have defined

$$\phi_n(Q) := -\frac{1}{2} \text{tr} \log Q + \frac{\beta \mu}{2} \text{tr} Q - \frac{\beta^2}{2} \sum_{a,b=1}^n B \left(\frac{Q_{aa} + Q_{bb} - 2Q_{ab}}{2} \right) \quad (53)$$

and

$$\mathcal{L}_Q(p) := \frac{i(\lambda - \mu_{eff})}{2} p + \frac{B''(0)}{4} p^2 - \frac{1}{2} \ln p \quad (54)$$

see [62].

where μ_{eff} is defined via

$$\mu_{eff} := \mu - \beta \sum_{a=1}^n B' \left(\frac{Q_{aa} + Q_{11} - Q_{a1}}{2} \right) \quad (55)$$

We can now change the order of integrations over P_d and Q and consider first the integration over $P_d = \text{diag}(p_1, \dots, p_m)$ for a fixed value of Q , i.e. find the stationary point w.r.t. P at fixed Q . The saddle point condition in p_α yields for all $\alpha = 1, \dots, m$

$$\begin{aligned} & \frac{\partial}{\partial p_\alpha} \left[\mathcal{L}_Q(p_\alpha) + \frac{B''(0)}{8} \left(\sum_{\alpha=1}^m p_\alpha \right)^2 \right] \\ &= \frac{1}{2} \left[i(\lambda - \mu_{eff}) + B''(0) \left(p_\alpha + \frac{1}{2} \sum_{\alpha} p_\alpha \right) - \frac{1}{p_\alpha} \right] = 0 \end{aligned} \quad (56)$$

We should now remember that (i) we are interested in the limit where λ is real ($\text{Im } \lambda = 0^-$) and (ii) we have the original constraint $p_\alpha > 0$, $\forall \alpha = 1, \dots, m$. The necessity to deform the integration contour to pass correctly through the stationary points then makes inavoiding to choose all p_α equal and such that $\text{Re } p_\alpha > 0$ (see e.g. [75] for a discussion) so that the only contributing saddle-point is independent of the index α :

$$p_\alpha = \frac{-i(\lambda - \mu_{eff}) + \sqrt{4B''_m(0) - (\lambda - \mu_{eff})^2}}{2B''_m(0)} := p_+^{(m)}, \quad \forall \alpha = 1, \dots, m \quad (57)$$

where we denoted $B''_m(0) := B''(0) \left(1 + \frac{m}{2}\right)$. Note that in the replica limit $m \rightarrow 0$ we have simply $B''_m(0) = B''(0)$.

At the next step we see that in the action Eq.(52) taken at the saddle point for P , the term $\sum_{\alpha=1}^m \mathcal{L}_Q(p_\alpha) = m \mathcal{L}_Q(p_+^{(m)})$ is proportional to m , and the last term in Eq.(52) is proportional to m^2 . If we now look for the saddle point for Q , and we consider m tending to zero before n does, the only term which couples P and Q , $m \mathcal{L}_Q(p_+) + \frac{B''(0)}{8} m^2 (p_+^{(m)})^2$, cannot modify the saddle-point value for the matrix Q . The latter therefore should be evaluated by extremizing $\phi_n(Q)$ only. We thus see that the two extremization procedures essentially decouple, which allows to perform the two replica limits $m \rightarrow 0$ and $n \rightarrow 0$ essentially independently. The same conclusion is reached if one solves the coupled saddle point equations for P, Q in the limit $m \rightarrow 0$, rather than sequentially as above.

From Eq.(45) and the above saddle point analysis we now conclude that

$$\lim_{N \rightarrow \infty} \overline{\langle G(\lambda, \mathbf{u}) \rangle}_T = i p_+^{(m=0)} \quad (58)$$

Using Eq.(13) this allows to extract the mean spectral density of the Hessian eigenvalues at a temperature T , and we find

$$\rho_T(\lambda) = 8 \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\pi(\lambda_+ - \lambda_-)^2}, \quad \int_{\lambda_-}^{\lambda_+} \rho_T(\lambda) d\lambda = 1 \quad (59)$$

where the lower and upper spectral edges given by

$$\lambda_- = \mu_{eff} - 2\sqrt{B''(0)}, \quad \lambda_+ = \mu_{eff} + 2\sqrt{B''(0)} \quad (60)$$

and μ_{eff} defined in Eq.(55). Hence the spectral density of the Hessian at the global minimum, $\lim_{T \rightarrow 0} \rho_T(\lambda)$ to leading order at large N , is a *semi-circle*, with two parameters, its center and its width. To determine those parameters one needs to solve the *equilibrium* optimization problem, by extremizing $\phi_n(Q)$ at a given inverse temperature β and eventually taking $n \rightarrow 0$. The resulting values for elements Q_{ab} of the optimal matrix Q in the limit $\beta \rightarrow \infty$ should be then substituted to Eqs.(55) and (60). Fortunately, the replica extremization problem arising here is exactly the same as solved in the various works described in Section 1.4 in the framework of the Parisi scheme of Replica Symmetry Breaking (RSB), and we will refer to mainly [56] and [50] when discussing this solution in the next section.

2.4. Solution of the extremization problem in the Parisi RSB scheme

The scheme starts with a standard assumption that in the replica limit $n \rightarrow 0$ the integral is dominated by configurations of matrices Q which for finite integer n have a special hierarchically built structure characterized by the sequence of integers

$$n = m_0 \geq m_1 \geq m_2 \geq \dots \geq m_k \geq m_{k+1} = 1 \quad (61)$$

and the values placed in the off-diagonal entries of the Q matrix block-wise, and satisfying:

$$0 < q_0 \leq q_1 \leq q_2 \leq \dots \leq q_k \quad (62)$$

Finally, we complete the procedure by filling in the n diagonal entries Q_{aa} of the matrix Q with one and the same value $Q_{aa} = q_d := q_{k+1} \geq q_k$. The subsequent treatment is much facilitated by introducing the following (generalized) function of the variable q :

$$x(q) = n + \sum_{l=0}^k (m_{l+1} - m_l) \theta(q - q_l) \quad (63)$$

where we use the notation $\theta(z)$ for the Heaviside step function: $\theta(z) = 1$ for $z > 0$ and zero otherwise. In view of the inequalities Eq.(61,62) the function $x(q)$ is piecewise-constant non-increasing, and changes between $x(0 < q < q_0) = m_0 \equiv n$ through $x(q_{i-1} < q < q_i) = m_i$ for $i = 1, \dots, k$ to finally $x(q_k < q < q_d) = m_{k+1} \equiv 1$. In the replica limit $n \rightarrow 0$ this function becomes a nontrivial non-increasing function of the variable $q \in [q_0, q_k]$, and outside that interval $x(0 < q < q_0) = 0$ and $x(q_k < q < q_d) = 1$. One can now express the eigenvalues of any function $g(Q)$ of the hierarchical matrix Q in terms of simple integrals involving $x(q)$ (see e.g. [42, 45, 46]) and eventually represent the replica limit of eq.(53) as a variational functional of $x(q)$ and the parameters q_d, q_0, q_k : $\lim_{n \rightarrow 0} \frac{1}{n} \phi_n(Q) = \phi_T[x(q); q_d, q_0, q_k]$. The explicit form of the functional is given, e.g., in Eq.(47) of [56] (we note that in the paper [56]

the covariance $B(q)$ is denoted $f(q)$, and the regularization parameter ϵ is denoted a^2). Similarly, it is easy to show that one can express the combinations entering Eq.(55) in terms of the same objects as

$$\mu_{eff} = \mu - \frac{1}{T} \left[B'(0) - B'(q_d - q_k) - \int_{q_0}^{q_k} B''(q_d - q)x(q)dq \right] \quad (64)$$

By varying the resulting functional $\phi_T[x(q); q_d, q_0, q_k]$ one obtains a system of stationarity equations, which may have different types of solutions which we briefly recapitulate below.

2.4.1. Replica-symmetric solution. In the range of parameters in the (μ, T) plane satisfying the inequality $\mu > \sqrt{B''(T/\mu)}$ the relevant stable solution to the stationarity equation is of the simplest, *replica-symmetric* form such that $q_0 = q_k$ so that the function $x(q)$ has no nontrivial support. The parameters q_k and q_d can be explicitly found:

$$q_d = \frac{T}{\mu} - \frac{1}{\mu^2} B' \left(\frac{T}{\mu} \right), \quad q_0 = -\frac{1}{\mu^2} B' \left(\frac{T}{\mu} \right) \quad (65)$$

In terms of them the combination μ_{eff} from Eq.(64) is given by

$$\mu_{eff} = \mu - \frac{1}{T} [B'(0) - B'(q_d - q_0)] \quad (66)$$

Combining Eq.(65) and Eq.(66) we see that μ_{eff} has a well-defined zero-temperature limit given by

$$\lim_{T \rightarrow 0} \mu_{eff} = \mu_{eff}^{(0)} = \mu + \frac{1}{\mu} B''(0) \quad (67)$$

This limit obviously makes sense only as long as it is performed in the phase where the RS solution remains stable at zero temperature, i.e. for μ satisfying $\mu > \mu_c = \sqrt{B''(0)}$. Substituting this value to Eq.(60) we find that the lowest edge of the semicircle is given by

$$\lambda_-^{(RS)} = \mu + \frac{1}{\mu} B''(0) - 2\sqrt{B''(0)} = \frac{(\mu - \mu_c)^2}{\mu} > 0, \quad \mu > \mu_c = \sqrt{B''(0)} \quad (68)$$

We conclude that the lowest eigenvalue of the Hessian at the global minimum in this case is separated by a finite gap. The width of this gap tends to zero when one approaches the critical value μ_c , known to separate the 'simple' type of landscape (i.e. one with typically a single minimum and no other saddle points) for $\mu > \mu_c$ from the 'complex' landscapes characterized by both exponentially many minima and saddle points of all indices [16, 19, 20] for $\mu < \mu_c$. Under these circumstances the replica-symmetric solution loses its stability via the so-called de Almeida-Thouless instability [76], and needs to be replaced by a different solution. The nature of the latter crucially depends on the type of correlations: short-ranged (SRC) vs. long-ranged (LRC).

2.4.2. 1-step replica symmetry breaking solution: SRC potentials. In the case of SRC potentials described in Eq.(20) the correct solution for $\mu < \mu_c$ corresponds to the simplest scheme of the replica symmetry breaking scheme, the so-called 1RSB, which is characterized by a step-like shape of the function $x(q)$:

$$x(q) = \begin{cases} 0, & q \in [0, q_0) \\ m, & q \in [q_0, q_1] \\ 1, & q \in (q_1, q_d] \end{cases} \quad (69)$$

where m is a temperature-dependent parameter to be found from the variation procedure (note that the parameter m used everywhere in this section should not be confused with the number of replica m used in the previous sections). Following [56] we find convenient to re-express the parameters q_1 and q_d in terms of combinations

$$\mathcal{Q} = q_1 - q_0, \quad y = q_d - q_1, \quad (70)$$

In these terms we find for μ_{eff} from Eq.(64) the following representation:

$$\mu_{eff} = \mu - \frac{1}{T} [B'(0) - B'(y)] + \frac{m}{T} [B'(y + \mathcal{Q}) - B'(y)] \quad (71)$$

Moreover, the stationarity conditions imply the relation $y = \frac{T}{\mu} - m \mathcal{Q}$ (see Eq.(74) in [56]). To perform the zero-temperature limit one first realizes that then $m \rightarrow 0$ in such a way that the ratio m/T tends to a finite limit, i.e. $\lim_{T \rightarrow 0} \frac{m}{T} = v < \infty$, then the parameter \mathcal{Q} tends to a finite value at $T = 0$.

Using these facts it is straightforward to perform the $T \rightarrow 0$ limit in Eq.(71) and obtain

$$\mu_{eff}^{(0)} = \mu + \frac{1}{\mu} B''(0) + v [B'(\mathcal{Q}) - B'(0) - B''(0) \mathcal{Q}] \quad (72)$$

so that for the lower edge of the Hessian spectrum we have

$$\lambda_-^{(1RSB)} = \frac{(\mu - \mu_c)^2}{\mu} + v (B'(\mathcal{Q}) - B'(0) - B''(0) \mathcal{Q}), \quad \mu < \mu_c = \sqrt{B''(0)} \quad (73)$$

Below we show in full generality that

$$\lambda_-^{(1RSB)} \geq 0 \quad (74)$$

Note that it is not immediately obvious to see that Eq.(73) satisfies Eq.(74) due to $v > 0$ and $\Delta := B'(\mathcal{Q}) - B'(0) - B''(0)\mathcal{Q} < 0$, $\forall \mathcal{Q} > 0$ for SRC covariances. As an example one can consider $B(\mathcal{Q}) = e^{-\mathcal{Q}}$ for which $\Delta = 1 - \mathcal{Q} - e^{-\mathcal{Q}} < 0$ for $\mathcal{Q} > 0$. To ascertain the sign of $\lambda_-^{(1RSB)}$ one needs to study the system of two closed equations satisfied by v and \mathcal{Q} everywhere for $\mu < \mu_c$ (Eqs.(89)-(90) in Eq. [56]), presented in the Introduction, see Eq.(7) and Eq.(8).

Now one can use Eq.(7) to eliminate $B'(\mathcal{Q}) - B'(0)$ in Eq.(73). After some rearranging one finds

$$\lambda_-^{(1RSB)} = \frac{\mu}{1 - \mu v \mathcal{Q}} + \frac{1 - \mu v \mathcal{Q}}{\mu} B''(0) - 2\sqrt{B''(0)} \quad (75)$$

$$= \left(\sqrt{\frac{\mu}{1 - \mu v \mathcal{Q}}} - \sqrt{\frac{1 - \mu v \mathcal{Q}}{\mu}} B''(0) \right)^2 = \frac{\mu_c^2}{\mu(1 - \mu v \mathcal{Q})} D^2 \geq 0 \quad (76)$$

where we denoted

$$D = \frac{\mu}{\mu_c} - 1 + \mu v \mathcal{Q} \quad (77)$$

The above expression can be used to study the critical behaviour of the spectral gap $\lambda_-^{(1RSB)}$ just below the threshold $\mu = \mu_c$. Note that the system of equations (7)-(8) is obviously always satisfied by $\mathcal{Q} = 0$ corresponding to the replica-symmetric solution, but we are interested in finding a nontrivial solution with $\mathcal{Q} > 0$. For a general $\mu < \mu_c$, and general $B(q)$, this can be done only numerically: see Fig. 1 in the Introduction for a plot of the lower edge of the Hessian for one example of a SRC potential. Note that we assume that $\mathcal{Q} > 0$ solution smoothly bifurcates from $\mathcal{Q} = 0$ at $\mu = \mu_c$, and follow this branch. In general one can not exclude a "first-order" type transition where in some range of parameters another solution with $\mathcal{Q} > 0$ appears discontinuously. A numerical solution of the equations however does not seem to support this mechanism.

Close to the critical value when $\delta = 1 - \frac{\mu}{\mu_c} \ll 1$ one can develop the perturbation theory in $0 < \delta \ll 1$. Since for $\delta \rightarrow 0$ we must have $\mathcal{Q} \rightarrow 0$ we look for a solution in the form:

$$v = v_0 + \delta v_1 + \delta^2 v_2 + \dots, \quad \mathcal{Q} = Q_1 \delta + Q_2 \delta^2 + Q_3 \delta^3 + \dots, \quad \text{for } \mu = \mu_c(1 - \delta) \quad (78)$$

where $\mu_c = \sqrt{B''(0)}$. To find these expansion coefficients we substitute Eq.(78) into Eqs.(7)-(8) and get for the two leading orders:

$$v_0 = -\frac{B'''(0)}{2\mu_c^3}, \quad Q_1 = -\frac{2\mu_c^2}{B'''(0)} \quad (79)$$

$$v_1 = -\frac{\frac{3}{2}[B'''(0)]^2 - B''(0)B''''(0)}{2\mu_c^3 B'''(0)}, \quad Q_2 = B''(0) \frac{3[B'''(0)]^2 - 10B''(0)B''''(0)}{6[B'''(0)]^3} \quad (80)$$

Substituting the expansion Eq.(78) into Eq.(77) we see that

$$D = -\delta \left[1 - \mu_c v_0 Q_1 \left(1 + \delta \left\{ \frac{v_1}{v_0} + \frac{Q_2 - Q_1}{Q_1} \right\} \right) \right] + O(\delta^3) \quad (81)$$

Now, Eq.(79) implies $\mu_c v_0 Q_1 = 1$ and further using Eq.(80) we arrive at

$$D = \frac{1}{6} \frac{\frac{3}{2}[B'''(0)]^2 - B''(0)B''''(0)}{[B'''(0)]^2} \delta^2 + O(\delta^3) \quad (82)$$

In view of Eq.(76) we then conclude that the first nonvanishing contribution to $\lambda_-^{(1RSB)}$ is order $O(\delta^4)$, and is explicitly given by Eq.(10) presented in the Introduction.

Moreover, using the discussion after Eq.(23) we see that the leading term in Eq.(10) is *strictly positive* in the case of all short-range potentials, and vanishes only for the logarithmically-correlated case Eq.(24). We will show in the end of our consideration, see Eq. (89) below, that such vanishing is not a coincidence: for the logarithmic case the spectral gap $\lambda_-^{(1RSB)}$ is *identically zero* everywhere in the phase with broken replica symmetry, $\mu < \mu_c$, so that the associated spectrum is *gapless*. In contrast, for a generic SRC potential the spectrum in the 1RSB phase is gapped, and tends to zero as proportional to $(\mu_c - \mu)^4$ on approaching the transition point $\mu_c = \sqrt{B''(0)}$, which is much faster than from the RS side, when vanishing is proportional to $(\mu - \mu_c)^2$, see Fig. 1.

2.4.3. Full replica symmetry breaking solution: LRC potentials. LRC potentials characterized by Eq.(21) define the most complex type of the Gaussian landscapes and are described by a pattern of full replica symmetry breaking (FRSB), with a nontrivial continuous non-increasing function $x(q)$ in the whole nonvanishing interval $q \in [q_0, q_k]$. This function can be found explicitly (see Eq.(53) of [56]):

$$x(q) = \begin{cases} 0, & q \in [0, q_0) \\ -\frac{T}{2} \frac{B'''(q_d - q)}{[B''(q_d - q)]^{3/2}}, & q \in [q_0, q_k] \\ 1, & q \in (q_k, q_d] \end{cases} \quad (83)$$

which allows to represent the parameter μ_{eff} in Eq.(64) in the form

$$\mu_{eff} = \mu - \sqrt{B''(q_d - q_0)} + \sqrt{B''(q_d - q_k)} - \frac{1}{T} [B'(0) - B'(q_d - q_k)] \quad (84)$$

It can be further shown that the parameters q_0, q_k and q_d in FRSB phase satisfy the following relations (see Eqs.(54) & (56) in [56])

$$q_d - q_k = \frac{T}{\sqrt{B''(q_d - q_k)}}, \quad \mu^2 = B''(q_d - q_0) \quad (85)$$

In the low-temperature limit $q_d - q_k \approx T/\sqrt{B''(0)}$ implying that

$$\lim_{T \rightarrow 0} \frac{B'(0) - B'(q_d - q_k)}{T} = -\sqrt{B''(0)}$$

which in turn results in the zero-temperature expression

$$\mu_{eff}^{(0)} = \mu - \sqrt{B''(q_d - q_0)} + 2\sqrt{B''(0)} = 2\sqrt{B''(0)} \quad (86)$$

where we used second of Eq.(85). We immediately see from Eq. (60) that in such a situation the spectrum of the Hessian is *gapless* for any $\mu < \mu_c$:

$$\lambda_-^{(FRSB)} = \mu_{eff}^{(0)} - 2\sqrt{B''(0)} = 0, \quad \mu < \mu_c = \sqrt{B''(0)} \quad (87)$$

|| Note that in the paper [56] the long-ranged potentials were formally considered without the

Obviously, such conclusion corroborates with the standard picture of huge degeneracy in the ground state of FRSB phase, indicating presence of 'marginally stable' spatial directions where one can move away from the global minimum almost at no increase of the landscape height.

2.4.4. Logarithmically-correlated potential. Finally, let us consider the boundary case of logarithmically-correlated potential, Eq.(24) which although described by 1-step SRB, is expected to share many features with FRSB systems (by this reason this case is sometimes called the "marginal 1RSB"). Going back to 1RSB formulas, we note that the parameters v and Q characterizing the zero-temperature solution can be found explicitly (see Eq.(64) in [56]), for $\mu < \mu_c$

$$v = \frac{1}{g}, \quad Q = \frac{g}{\mu} - \epsilon \quad (88)$$

whereas $B'(q) = -g^2/(q+\epsilon)$ so that $B'(0) = -g^2/\epsilon$ and $B''(0) = g^2/\epsilon^2$. Substituting these values to Eq.(73) we see that in such a case

$$\lambda_-^{(\log)} = \begin{cases} \frac{(\mu - \mu_c)^2}{\mu} > 0, & \mu > \mu_c = \frac{g}{\epsilon} \\ 0, & \mu < \mu_c = \frac{g}{\epsilon} \end{cases} \quad (89)$$

exactly like in FRSB case. Note that the critical value of the confinement μ_c crucially depends on the short-ranged regularization $\epsilon > 0$.

3. Conclusions and Discussions

In this paper we adapted the replica trick to derive the mean eigenvalue density for the Hessian associated with the global minimum of one of the most paradigmatic models of high-dimensional landscapes: an isotropic homogeneous Gaussian-distributed random potential superimposed on a simple parabolic background with the curvature parameter $\mu > 0$, see Eq.(1). We showed that the Hessian eigenvalue density is always of the semicircular form. The position of the lowest edge of the semicircle was found to reflect the nature of the underlying potential controlled by both the value of the curvature μ of the deterministic part and of the covariance structure of the random part. Simple landscapes with generically a single minimum are typical for $\mu > \mu_c = \sqrt{B''(0)}$, and we show that for these landscapes the Hessian at the global minimum is always *gapped*, with the low spectral edge being strictly positive. The spectral gap vanishes as $(\mu - \mu_c)^2$ when approaching from above the transition point $\mu = \mu_c$ separating simple landscapes from "glassy" ones, with exponentially abundant minima. For $\mu < \mu_c$ the Hessian spectrum is qualitatively different for 'genuinely complex' and 'moderately complex' landscapes.

short-scale regularization, i.e. for $\epsilon = 0$. This resulted in $B''(0) = \infty$ and hence the replica-symmetry was fully broken at zero temperature at any value of μ . For the purposes of the present

In the "genuinely complex" landscapes with long-ranged power-law correlations the replica symmetry is completely broken and the Hessian remains gapless for all values of $\mu < \mu_c$, indicating presence of 'marginally stable' spatial directions, as was anticipated in other models with fully broken replica symmetry, see [33, 34] and a discussion in the introduction. In contrast, the 'moderately complex' landscapes are typical for short-range correlated random potentials and correspond to the 1-step replica-symmetry breaking mechanism. Their Hessian spectra turn out to be in general gapped, with the gap vanishing on approaching μ_c from below with a larger critical exponent, as $(\mu_c - \mu)^4$, see Eq.(10) and the Figure 1. To the best of our knowledge this type of critical behaviour has not yet been discussed previously. It would be therefore interesting to understand what are the associated changes of the landscape shape around the global minimum which is responsible for such an asymmetry of the lowest Hessian eigenvalue. Finally, we demonstrated that the potentials with *logarithmic* correlations, which are at the frontier between 1RSB and FRSB (they are called marginal 1RSB), share both the 1RSB nature and a gapless spectrum. It is natural to expect the main features revealed in the present paper to be generic for other 'topological trivialization' transitions associated with the 1RSB mechanism (e.g. one induced by increasing magnetic field in the spherical model, see e.g. [21, 23, 24] and references therein). We leave this and related problems for a future investigation.

Concerning the possibility of a direct numerical verification of the results of the paper one should have in mind that it is extremely challenging to generate the high-dimensional random fields which are both *stationary* and *isotropic*. To bypass this difficulty one may rely upon anticipated universality of the results and perform simulations of high-dimensional *isotropic* random fields confined to a sphere which can be efficiently generated using multivariate polynomials with random coefficients. The resulting model is equivalent to an instance of a 'spherical spin glass', with the role of the control parameter μ played by an external magnetic field, which has a similar "topology trivialization" effect. For such a model it is known to be possible, in dimensions of the order $N \sim 10^1$, to enumerate all stationary points and therefore select the global minimum with due efforts and then study the Hessian around it. Such simulations are computationally challenging but not completely prohibitive, see e.g. [77].

The method used here is quite versatile and opens the way to explore further questions. In particular, it should be possible to study 'bulk' spectral correlations (presumably, of Wigner-Dyson type), and further fluctuations of (delocalized) eigenvectors of the Hessian within a variant of our method involving 'superbosonisation' approach to resolvents [78, 79]. What concerns the few lowest eigenvalues of the Hessian, those would be natural to expect to show Tracy-Widom type fluctuations in the gapped phases, whereas at the transition point and in the gapless phase their statistics should be quite different. This will require studying study keeping a finite regularization is essential.

finite N corrections, and one may wonder what remains of the present picture at finite N , especially in the interesting case of logarithmic potentials where the RSB transition is known to survive (and for which, in fact, infinite and finite N share a lot of similarities) [62, 63, 80]. Obtaining a more detailed picture of the low lying states and their associated Hessian would also be of great interest. Finally, extensions to pinned manifolds embedded in high-dimensional spaces are presently under investigation [61].

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Appendix A. Appendix: $O(N)$ -invariant integration formula.

In this Appendix we recall some identities which are very helpful when dealing with integrals with $O(N)$ -invariant integrands. A statement equivalent to one of the Theorem I below seemingly first appeared in [81], and was later on rediscovered in [75]. A detailed proof is provided in Appendix B of [80] (see also Appendix A of [82]).

Theorem I

Consider a function $F(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of N -component real vectors \mathbf{u}_a with $1 \leq a \leq n$ such that

$$\int_{R^N} d\mathbf{u}_1 \dots \int_{R^N} d\mathbf{u}_n |F(\mathbf{u}_1, \dots, \mathbf{u}_n)| < \infty \quad (\text{A.1})$$

Denoting by \mathbf{u}^T the vector transposition suppose further that the function F is invariant with respect to any global $O(N)$ rotation: $\mathbf{u}_a \rightarrow O\mathbf{u}_a, \forall a = 1, \dots, n$. Then necessarily it depends only on $n(n+1)/2$ scalar products $\mathbf{u}_a^T \mathbf{u}_b$ $1 \leq a, b \leq n$ and can be therefore rewritten as a function $F(\mathbf{u}_1, \dots, \mathbf{u}_n) = \mathcal{F}(\tilde{Q})$ of the $n \times n$ real symmetric matrix \tilde{Q} :

$$\tilde{Q}_{a,b} = \mathbf{u}_a^T \mathbf{u}_b \mid_{1 \leq a, b \leq n}$$

Then for $N > n$ the integral defined as

$$I_{N,n} = \int_{R^N} d\mathbf{u}_1 \dots \int_{R^N} d\mathbf{u}_n F(\mathbf{u}_1, \dots, \mathbf{u}_n) \quad (\text{A.2})$$

is equal to

$$I_{N,n} = \mathcal{C}_{N,n}^{(o)} \int_{Q \geq 0} dQ (\det Q)^{(N-n-1)/2} \mathcal{F}(Q) \quad (\text{A.3})$$

where the proportionality constant is given by

$$\mathcal{C}_{N,n}^{(o)} = \frac{\pi^{\frac{n}{2}(N-\frac{n-1}{2})}}{\prod_{k=0}^{n-1} \Gamma\left(\frac{N-k}{2}\right)} \quad (\text{A.4})$$

and the integration in Eq.(A.3) goes over the manifold of real symmetric positive semidefinite $n \times n$ matrices $Q \geq 0$, with measure $dQ = \prod_{a \leq b} dQ_{ab}$.

One of the ways of proving the Theorem is to exploit the following matrix integral identity:

$$\mathcal{J}_{n,N,\epsilon}(Q) = \int e^{-\frac{i}{2}\text{tr}[QX]} \frac{1}{\det(\epsilon \mathbf{1}_n - iX)^{N/2}} dX, \quad (\text{A.5})$$

$$= \frac{1}{2^{\frac{Nn}{2}} \pi^{\frac{n(n-1)}{4}}} \frac{1}{\prod_{j=0}^{n-1} \Gamma\left(\frac{N-j}{2}\right)} \det Q^{\frac{N-n-1}{2}} e^{-\frac{1}{2}\epsilon \text{tr} Q} \prod_{j=1}^n \theta(q_j) \quad (\text{A.6})$$

where the integration goes over $n \times n$ real symmetric matrices: $X_{ab} = X_{ba}$ with the volume element $dX = \prod_a \frac{dX_{aa}}{4\pi} \prod_{a < b} \frac{dX_{ab}}{2\pi}$, $\epsilon > 0$, $N \geq n+1$, $\Gamma(\nu)$ is the Euler Gamma-function, Q is $n \times n$ real symmetric and $\theta(x) = 1$ for $x \geq 0$ and zero otherwise. The identity can be proved [75, 80] following a method suggested for evaluating a related integral due to Ingham and Siegel [83].

Finally, let us mention that there exists a useful generalization of the Theorem I, known as the 'superbosonisation', which extends it to the case when vectors \mathbf{u} are replaced with *supervectors* involving both commuting ('bosonic') and anticommuting ('fermionic') entries, see an informal account in [79].

Appendix B. Weak confinement limit $\mu \rightarrow 0$ for SRC potentials

Here we consider the case of SRC potentials, with $qB'(q) \rightarrow 0$ as $q \rightarrow +\infty$ which is generic but excludes the marginal case of logarithmically-correlated potential. Under this condition we establish the leading small μ behavior of the lowest Hessian edge.

In the limit $\mu \rightarrow 0$ the equations (7)-(8) lead to $\mathcal{Q}, v \rightarrow +\infty$, $1 - \mu v \mathcal{Q} \rightarrow 0$ with, keeping only the leading terms

$$\frac{\mu^2 \mathcal{Q}}{1 - \mu v \mathcal{Q}} \simeq -B'(0) \quad , \quad \ln(1 - \mu v \mathcal{Q}) \simeq -1 - B(0)v^2 \quad (\text{B.1})$$

Multiplying the first equation by v/μ , and denoting $X = 1 - \mu v \mathcal{Q}$ we obtain

$$\frac{1}{X} \simeq 1 - v \frac{B'(0)}{\mu} \quad , \quad \ln \frac{1}{X} = 1 + v^2 B(0) \quad (\text{B.2})$$

Introducing $y := v\sqrt{B(0)} \gg 1$ we then have from the above system:

$$y^2 \approx \ln \frac{1}{e\mu} + \ln y + \ln \frac{-B'(0)}{\sqrt{B(0)}} + O(1/y) \quad (\text{B.3})$$

which is solved asymptotically for $\mu \rightarrow 0$ by

$$y \approx \sqrt{\ln \frac{1}{e\mu}} + \frac{1}{4} \frac{\ln \ln \frac{1}{e\mu}}{\sqrt{\ln \frac{1}{e\mu}}} \quad (\text{B.4})$$

which finally gives for the gap from Eq.(75)

$$\begin{aligned}\lambda_-^{(1\text{RSB})} &= \frac{\mu}{X} + \frac{X}{\mu} B''(0) - 2\sqrt{B''(0)} \simeq -\frac{B'(0)}{\sqrt{B(0)}} y - 2\sqrt{B''(0)} \\ &\simeq \frac{-B'(0)}{\sqrt{B(0)}} \left[\sqrt{\ln \frac{1}{e\mu}} + \frac{1}{4} \frac{\ln \ln \frac{1}{e\mu}}{\sqrt{\ln \frac{1}{e\mu}}} \right] - 2\sqrt{B''(0)}\end{aligned}\quad (\text{B.5})$$

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